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Optimum Sensitivity-Based Statistical Parameters Estimation from Modal Response

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Based on the concept of optimum sensitivity, we present a method for estimating the mean and covariance of parameters of a mechanical system from the statistics of its measured modal response. The optimum sensitivity, defined as the sensitivity of system parameters with respect to observed output, is obtained by direct differentiation of the Kuhn–Tucker optimality criteria for a nonlinear least-squares output error estimator. With the optimum sensitivity derivatives up to the second order, we can estimate the second-order approximation of both the mean and covariance of the system parameters by applying methods developed originally for evaluating the output of uncertain systems based on the more conventional notion of sensitivity, the sensitivity of system response with respect to system parameters. The present approach allows us to assess the bias due to nonlinearities in the least-squares estimator whereas conventional sensitivity-based methods do not. Furthermore, the present method is generally much more efficient than Monte Carlo simulation because nonlinear optimization is performed only once. We demonstrate through example problems that, compared to the conventional sensitivity-based methods, the present method provides statistical indices that are more consistent with those obtained by Monte Carlo simulation.

Nomenclature

a = level of uncertainty in measurement
$\boldsymbol{e}$ = error function
J = objective function
$\boldsymbol{K}$ = stiffness matrix
$\boldsymbol{M}$ = mass matrix
N_D = number of degrees of freedom
N_L = number of measured locations
N_M = number of measured modes
N_P = number of system parameters
$\boldsymbol{R}^{x^*}$ = covariance of estimate vectors
$\boldsymbol{R}^{\Phi}$ = covariance of measurement vectors
$\boldsymbol{x}$ = system parameters
$\boldsymbol{x}^N$ = nominal value of system parameters
$\boldsymbol{x}^*$ = estimate of system parameters
$\bar{\boldsymbol{x}}^*$ = mean of estimates of system parameters
α_m = mth modal weighting factor
$\hat{\boldsymbol{\Phi}}$ = measurement vector
$\bar{\boldsymbol{\Phi}}$ = mean of measurement vectors
$\hat{\Phi}_\mu$ = μth component of $\hat{\boldsymbol{\Phi}}$
ϕ_m = mth eigenvector
ω_m = mth natural frequency
$\nabla()$ = gradient with respect to $\boldsymbol{x}$
$\nabla_i()$ = partial differentiation with respect to x_i

Subscript

$,\mu$ = partial differentiation with respect to $\hat{\Phi}_\mu$

I. Introduction

UNCERTAINTY, both in observations and in system parameters, plays a significant role in parameter estimation, wherein system parameters are estimated by minimizing the difference between observations and predictions of a model based on the system parameters to be estimated.[1,2] Many researchers have recognized that a statistical approach is needed for this class of problems. In this paper, we shall refer to the following problem as statistical parameter estimation: Given the probability distribution of a set of observations, find certain statistical indices of the system parameters. The basic framework of statistical parameter estimation is schematically illustrated in Fig. 1a. This problem is important when, for instance, we assess the bias and covariance of estimates, which can be used to find the best measurement locations. This problem is different from that solved in the Bayesian framework or the maximum likehood method, in which one usually obtains a unique estimate as the most likely value of the system parameters from a single measured data set, knowing probability distribution of system response given the system parameters.

Parameter estimation is generally cast as least-squares error minimization. If the error function is linear in the system parameters, the least-squares minimization is a quadratic programming problem, and a closed-form expression of an estimate of system parameters is possible. Moreover, if the error function is linear with respect not only to system parameters, but also to measured outputs, we can obtain closed-form expressions for the statistical indices of the system parameters in terms of the statistical indices of the measurements. On the other hand, if the error function is nonlinear, iterative procedures are generally required to solve the nonlinear optimization problem. Many parameter estimation problems in the field of structural mechanics fall into this class of nonlinear least-squares error minimization problems.[3–7]

To obtain the statistical indices of system parameters in the nonlinear least-squares minimization, Monte Carlo simulation has been used frequently in the literature.[1,3,8] In a Monte Carlo simulation, synthetic observations $\hat{\boldsymbol{\Phi}}_i$ $(i = 1, 2, \ldots)$ are created by using computer-generated random numbers such that the sample has a specified probability distribution as shown in Fig. 1b. Next, a set of parameter estimates $\boldsymbol{x}_i^*$ are obtained by repeating the nonlinear least-squares minimization for each artificial observation. Approximate statistical indices of the set of estimated system parameters can be obtained according to the standard definition of the statistical indices of a finite discrete data set. Thus, the basic concept behind the Monte Carlo simulation is simple and straightforward. The approximations improve as the number of synthetic measurements increases. The Monte Carlo simulation is, however, often computationally expensive because nonlinear optimization is repeated many times.

In this paper, based on the concept of optimum sensitivity, we present a statistical parameter estimation method for a nonlinear least-squares estimator. Throughout this paper, we concentrate on parameter estimation from modal response. In this problem,

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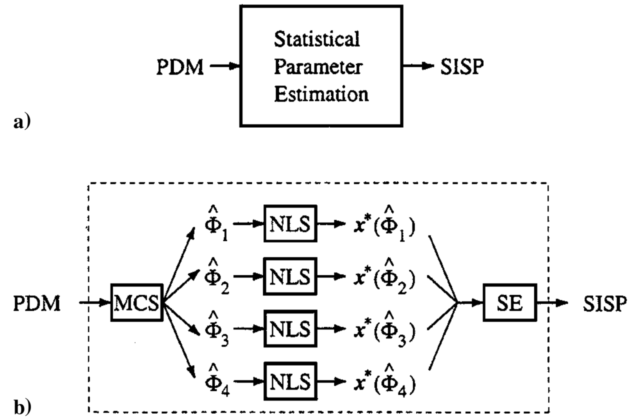


Fig. 1 Basic concepts of a) statistical parameter estimation and b) Monte Carlo simulation [probability distribution of measurements (PDM), statistical indices of system parameters (SISP), nonlinear least-squares minimization (NLS), Monte Carlo simulation (MCS), statistical evaluation (SE)].

optimum sensitivity can be viewed as the sensitivity of system parameters with respect to system outputs, the eigenvalues and eigenvectors of the free vibration problem, which can be measured in a number of different ways. It should be evident that the ideas can be easily extended to static and transient dynamic parameter estimation. The notion of optimum sensitivity used here is conceptually different from, although closely related to, that of conventional sensitivity, the sensitivity of the system response with respect to the system parameters. Following the approach outlined by Sobieszczanski-Sobieski et al.[9] and Schmit and Chang,[10] we obtain optimum sensitivity derivatives by direct differentiation of the Kuhn–Tucker optimality criteria. This approach is generally more robust with respect to roundoff errors and more efficient than the finite difference method used by Sanayei et al.[8] From the viewpoint of statistics, optimum sensitivity derivatives enable us to apply efficient methods for evaluating the statistics of vector-valued functions,[11] which were originally developed for evaluating the output of uncertain systems based on conventional sensitivity information. We employ the perturbation method[12] because of its simplicity and efficiency.

The organization of this paper is as follows. Section II presents the basic equations for parameter estimation from modal response. Section III describes the formulation of optimum sensitivity. In Sec. IV, based on the perturbation method, we derive the formulas for statistical parameter estimation using the optimum sensitivity derivatives. In Sec. V, we show that the present method can be viewed as a generalization of the statistical parameter estimation methods that use the conventional sensitivity. In Sec. VI, we examine the performance and limitations of the present method via two example problems.

II. Parameter Estimation from Modal Response

The basic equations are given in this section for parameter estimation of a structure from modal response. We make the following assumptions on structural modeling although more general modeling is possible[13,14]: 1) The structure is discretized with a finite element model with N_D degrees of freedom (DOF). 2) The stiffness matrix $\boldsymbol{K}$ is parameterized by a system parameter vector $\boldsymbol{x}$ with N_P components. 3) All of the system parameters but $\boldsymbol{x}$ are known and deterministic. 4) The structure is undamped. 5) Structural response is linear with respect to external loads. With these assumptions, our objective is to estimate $\boldsymbol{x}$. The eigenvalue problem associated with free (undamped) vibration of the structure can be expressed as

$$\boldsymbol{K}(\boldsymbol{x})\phi_m = \omega_m^2 \boldsymbol{M}\phi_m \qquad (m = 1, 2, \ldots, N_D) \tag{1}$$

where ω_m is the mth natural frequency and ϕ_m is the eigenvector corresponding to ω_m. Throughout this paper, ϕ_m is normalized with $\phi_m^{\max} = 1$, where $\phi_m^{\max}$ is the component that has the maximum absolute value in ϕ_m. For simple presentation of our method, we make the following assumptions concerning the observations: 1) $N_M(\leq N_D)$ modes are measured at $N_L(\leq N_D)$ discrete locations. 2) The total number of measurements exceeds the number of system parameters, that is, $N_L N_M \geq N_P$. 3) We consider uncertainty only in mode shape measurements, and we do not consider uncertainty in measured frequencies. Note, however, that it is straightforward to incorporate the uncertainty of frequencies. The quantities corresponding to the measured and unmeasured locations are indicated by carat and tilde, respectively. To wit, $\hat{\phi}_m$ and $\tilde{\phi}_m$ refer to the parts of ϕ_m associated with the measured and unmeasured locations, respectively. Let a superscript T indicate the transpose of a vector or a tensor. Let the eigenvector ϕ_m be partitioned as $\phi_m^T = \{\hat{\phi}_m^T | \tilde{\phi}_m^T\}$. Define a measurement vector $\hat{\boldsymbol{\Phi}}$ of $N_L N_M$ components as

$$\hat{\boldsymbol{\Phi}}^T \equiv \left\{\hat{\phi}_1^T, \hat{\phi}_2^T, \ldots, \hat{\phi}_{N_M}^T\right\} \tag{2}$$

From the measurement $\hat{\boldsymbol{\Phi}}$, we estimate the system parameter vector $\boldsymbol{x}$ as the solution of the constrained minimization problem

$$\min_{\boldsymbol{x}} J(\boldsymbol{x}, \hat{\boldsymbol{\Phi}}) \tag{3}$$

subject to $\boldsymbol{c}_L \leq \boldsymbol{x} \leq \boldsymbol{c}_U$, where $\boldsymbol{c}_L$ and $\boldsymbol{c}_U$ are the lower and upper bounds of $\boldsymbol{x}$ specified prior to the optimization. With modal weighting factors α_m, we define the least-squares objective function J as

$$J \equiv \frac{1}{2}\sum_{m=1}^{N_M} \alpha_m \boldsymbol{e}(\boldsymbol{x}, \hat{\phi}_m) \cdot \boldsymbol{e}(\boldsymbol{x}, \hat{\phi}_m) \tag{4}$$

where the center dot indicates the vector inner product. Let the mass matrix $\boldsymbol{M}$ be partitioned with columns associated with measured $(\hat{\phi})$ and unmeasured $(\tilde{\phi})$ DOF $\boldsymbol{M} = [\hat{\boldsymbol{M}} \,|\, \tilde{\boldsymbol{M}}]$, where $\hat{\boldsymbol{M}}$ is $N_D \times N_L$ and $\tilde{\boldsymbol{M}}$ is $N_D \times (N_D - N_L)$ in size. Let the stiffness matrix $\boldsymbol{K}$ be partitioned similarly. Furthermore, define $\boldsymbol{M}_0 \equiv [\hat{\boldsymbol{0}} \mid \tilde{\boldsymbol{M}}]$ to be the mass matrix with the $\hat{\boldsymbol{M}}$ part zeroed out, that is, $\hat{\boldsymbol{0}}$ is an $N_D \times N_L$ zero matrix. With these definitions, we define the error function $\boldsymbol{e}_m$ as

$$\boldsymbol{e}_m(\boldsymbol{x}, \hat{\phi}_m) \equiv \hat{\phi}_m - \omega_m^2 \boldsymbol{Q}\boldsymbol{B}_m^{-1}(\boldsymbol{x})\hat{\boldsymbol{M}}\hat{\phi}_m \tag{5}$$

Here $\boldsymbol{Q}$ is a boolean matrix and can be defined as $\boldsymbol{Q} \equiv \partial\hat{\phi}_1/\partial\phi_1$, which is the same for each mode, and $\boldsymbol{B}_m(\boldsymbol{x}) \equiv \boldsymbol{K}(\boldsymbol{x}) - \omega_m^2 \boldsymbol{M}_0$. As demonstrated by Hjelmstad et al.,[14] the error function $\boldsymbol{e}_m$ is identically zero when the mode shapes and frequency are the same as those gained by Eq. (1) for given $\boldsymbol{x}$. Note that we deliberately avoid the use of regularization, or penalty, methods in this paper because it is often difficult to set appropriate penalty parameters. It is, however, straightforward to incorporate them.

The parameter estimates can be found using a standard numerical technique for constrained nonlinear optimization.[13,14] Define the Lagrangian function L as

$$L(\boldsymbol{x}, \boldsymbol{\lambda}, \hat{\boldsymbol{\Phi}}) \equiv J(\boldsymbol{x}, \hat{\boldsymbol{\Phi}}) + \boldsymbol{\lambda}_L \cdot (\boldsymbol{c}_L - \boldsymbol{x}) + \boldsymbol{\lambda}_U \cdot (\boldsymbol{x} - \boldsymbol{c}_U) \tag{6}$$

where $\boldsymbol{\lambda}_L$ and $\boldsymbol{\lambda}_U$ are the vector of the Lagrangian multipliers corresponding to the lower and upper bounds, respectively. Let ∇ denote the gradient with respect to $\boldsymbol{x}$. Then the optimal solution $\boldsymbol{x}^*$ and $\boldsymbol{\lambda}^*$ satisfy the Kuhn–Tucker conditions

$$\begin{gathered}\nabla L(\boldsymbol{x}^*) = \nabla J(\boldsymbol{x}^*) - \boldsymbol{\lambda}_L^* + \boldsymbol{\lambda}_U^* = \boldsymbol{0}, \qquad \boldsymbol{\lambda}_L^*\colon \left(\boldsymbol{c}_L - \boldsymbol{x}^*\right) = \boldsymbol{0} \\ \boldsymbol{\lambda}_U^*\colon \left(\boldsymbol{x}^* - \boldsymbol{c}_U\right) = \boldsymbol{0}, \qquad \boldsymbol{x}^* - \boldsymbol{c}_L \geq \boldsymbol{0}, \qquad \boldsymbol{c}_U - \boldsymbol{x}^* \geq \boldsymbol{0} \\ \boldsymbol{\lambda}_L^* \geq \boldsymbol{0}, \qquad \boldsymbol{\lambda}_U^* \geq \boldsymbol{0}\end{gathered} \tag{7}$$

where colon indicates the component by component multiplication of vectors. For active subsets $\boldsymbol{\lambda}_L^{*A}$ and $\boldsymbol{\lambda}_U^{*A}$ of the Lagrangian multipliers $\boldsymbol{\lambda}_L^*$ and $\boldsymbol{\lambda}_U^*$, Eq. (7) can be rewritten as

$$\begin{gathered}\nabla J(\boldsymbol{x}^*) - \boldsymbol{Q}_L^T \boldsymbol{\lambda}_L^{*A} + \boldsymbol{Q}_U^T \boldsymbol{\lambda}_U^{*A} = \boldsymbol{0}, \qquad \boldsymbol{Q}_L\left(\boldsymbol{x}^* - \boldsymbol{c}_L\right) = \boldsymbol{0} \\ \boldsymbol{Q}_U\left(\boldsymbol{x}^* - \boldsymbol{c}_U\right) = \boldsymbol{0}\end{gathered} \tag{8}$$

where the boolean matrices $\boldsymbol{Q}_L$ and $\boldsymbol{Q}_U$ extract the sets of active constraints for the lower and upper bounds, respectively.

III. Optimum Sensitivity

Following the approach of Sobieszczanski-Sobieski et al.[9] and Schmit and Chang,[10] we present the formulation for obtaining optimum sensitivity, the sensitivity of the estimate $\boldsymbol{x}^*$ with respect to the observation $\hat{\boldsymbol{\Phi}}$. Suppose that we have the solution of the minimization problem (3). Then, noticing that all $\boldsymbol{x}^*$, $\boldsymbol{\lambda}_L^{*A}$, and $\boldsymbol{\lambda}_U^{*A}$ are functions of $\hat{\boldsymbol{\Phi}}$, we can obtain the optimum sensitivity by direct differentiation of the Kuhn–Tucker condition (8) with respect to $\hat{\boldsymbol{\Phi}}$. Let $\hat{\Phi}_\mu$ denote the μth component of $\hat{\boldsymbol{\Phi}}$, and let the optimum sensitivity derivatives be indicated by

$$\boldsymbol{x}^*_{,\mu} \equiv \frac{\partial \boldsymbol{x}^*}{\partial \hat{\Phi}_\mu}, \qquad \boldsymbol{x}^*_{,\mu\nu} \equiv \frac{\partial^2 \boldsymbol{x}^*}{\partial \hat{\Phi}_\mu \partial \hat{\Phi}_\nu} \tag{9}$$

Define differential operators in component form as

$$\nabla_i(\,) \equiv \frac{\partial(\,)}{\partial x_i}, \qquad \nabla^2_{ij}(\,) \equiv \frac{\partial^2(\,)}{\partial x_i \partial x_j}$$

$$\nabla^3_{ijk}(\,) \equiv \frac{\partial^3(\,)}{\partial x_i \partial x_j \partial x_k} \tag{10}$$

where all of the subscripts i, j, k range from 1 to N_P. Differentiating Eq. (8) with respect to $\hat{\Phi}_\mu$, we have

$$\begin{bmatrix} \nabla^2 J & -\boldsymbol{Q}_L^T & \boldsymbol{Q}_U^T \\ -\boldsymbol{Q}_L & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{Q}_U & \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{x}^*_{,\mu} \\ \boldsymbol{\lambda}^{*A}_{L,\mu} \\ \boldsymbol{\lambda}^{*A}_{U,\mu} \end{bmatrix} = \begin{bmatrix} \boldsymbol{C}^1_\mu \\ \boldsymbol{0} \\ \boldsymbol{0} \end{bmatrix} \tag{11}$$

where the coefficient matrix $\nabla^2 J$ and the known vector $\boldsymbol{C}^1_\mu$ are obtained in component form as

$$\nabla^2_{ij} J = \sum_{m=1}^{N_M} \alpha_m \left(\boldsymbol{e}_m \cdot \nabla^2_{ij} \boldsymbol{e}_m + \nabla_i \boldsymbol{e}_m \cdot \nabla_j \boldsymbol{e}_m \right) \tag{12}$$

$$\left[\boldsymbol{C}^1_\mu\right]_i \equiv -\nabla_i J_{,\mu} = -\sum_{m=1}^{N_M} \alpha_m (\boldsymbol{e}_m \cdot \nabla_i \boldsymbol{e}_{m,\mu} + \nabla_i \boldsymbol{e}_m \cdot \boldsymbol{e}_{m,\mu}) \tag{13}$$

When the number of the active constraints is small, this simultaneous equation can be solved efficiently using the bordering algorithm.[15] All of the derivatives of $\boldsymbol{e}_m$ that appear in this section are given in detail in Appendix A. By solving Eq. (11) for each index $\mu (\mu = 1, \ldots, N_L N_M)$, we can obtain the first-order optimum sensitivity derivatives $\boldsymbol{x}^*_{,\mu}$, $\boldsymbol{\lambda}^{*A}_{L,\mu}$, and $\boldsymbol{\lambda}^{*A}_{U,\mu}$. Note that $\nabla^2 J$ is same for every μ and needs to be factored only once.

We can obtain higher-order derivatives by repeated differentiation of the Kuhn–Tucker conditions. For instance, differentiation of Eq. (11) with respect to $\hat{\Phi}_\nu$ yields the equations for the second-order optimum sensitivity derivatives $\boldsymbol{x}^*_{,\mu\nu}$, $\boldsymbol{\lambda}^*_{L,\mu\nu}$, and $\boldsymbol{\lambda}^*_{U,\mu\nu}$ as

$$\begin{bmatrix} \nabla J^2 & -\boldsymbol{Q}_L^T & \boldsymbol{Q}_U^T \\ -\boldsymbol{Q}_L & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{Q}_U & \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{x}^*_{,\mu\nu} \\ \boldsymbol{\lambda}^{*A}_{L,\mu\nu} \\ \boldsymbol{\lambda}^{*A}_{U,\mu\nu} \end{bmatrix} = \begin{bmatrix} \boldsymbol{C}^2_{\mu\nu} \\ \boldsymbol{0} \\ \boldsymbol{0} \end{bmatrix} \tag{14}$$

where the known vector $\boldsymbol{C}^2_{\mu\nu}$ in Eq. (14) can be expressed as

$$\left[\boldsymbol{C}^2_{\mu\nu}\right]_i \equiv -\boldsymbol{x}^*_{,\mu} \cdot \boldsymbol{Z}_i \boldsymbol{x}^*_{,\nu} - \left[\nabla^2 J_{,\mu} \boldsymbol{x}^*_{,\nu}\right]_i - \left[\nabla^2 J_{,\nu} \boldsymbol{x}^*_{,\mu}\right]_i - \nabla_i J_{,\mu\nu} \tag{15}$$

The terms required to evaluate this expression can be computed as

$$[\boldsymbol{Z}_i]_{jk} \equiv \nabla^3_{ijk} J = \sum_{m=1}^{N_M} \alpha_m \Big(\boldsymbol{e}_m \cdot \nabla^3_{ijk} \boldsymbol{e}_m + \nabla_i \boldsymbol{e}_m \cdot \nabla^2_{jk} \boldsymbol{e}_m + \nabla_j \boldsymbol{e}_m \cdot \nabla^2_{ki} \boldsymbol{e}_m + \nabla_k \boldsymbol{e}_m \cdot \nabla^2_{ij} \boldsymbol{e}_m \Big)$$

$$\nabla^2_{ij} J_{,\mu} = \sum_{m=1}^{N_M} \alpha_m \Big(\boldsymbol{e}_m \cdot \nabla^2_{ij} \boldsymbol{e}_{m,\mu} + \nabla_i \boldsymbol{e}_m \cdot \nabla_j \boldsymbol{e}_{m,\mu} + \nabla_j \boldsymbol{e}_m \cdot \nabla_i \boldsymbol{e}_{m,\mu} + \boldsymbol{e}_{m,\mu} \cdot \nabla^2_{ij} \boldsymbol{e}_m \Big)$$

$$\nabla_i J_{,\mu\nu} = \sum_{m=1}^{N_M} \alpha_m (\boldsymbol{e}_m \cdot \nabla_i \boldsymbol{e}_{m,\mu\nu} + \nabla_i \boldsymbol{e}_m \cdot \boldsymbol{e}_{m,\mu\nu} + \boldsymbol{e}_{m,\mu} \cdot \nabla_i \boldsymbol{e}_{m,\nu} + \boldsymbol{e}_{m,\nu} \cdot \nabla_i \boldsymbol{e}_{m,\mu}) \tag{16}$$

Note again that $\nabla^2 J$ need not be factored again in the computation of the second-order optimum sensitivity derivatives because it is the same as that for obtaining the first-order derivatives.

IV. Statistical Parameter Estimation

Based on the perturbation method,[11,12] we derive the formula for the approximate statistical indices of the system parameters. Suppose that we have the mean and covariance of the measurements defined, respectively, as

$$\bar{\boldsymbol{\Phi}} \equiv E[\hat{\boldsymbol{\Phi}}] \tag{17}$$

$$\boldsymbol{R}^\Phi \equiv E[(\hat{\boldsymbol{\Phi}} - \bar{\boldsymbol{\Phi}}) \otimes (\hat{\boldsymbol{\Phi}} - \bar{\boldsymbol{\Phi}})] \tag{18}$$

where $\otimes$ indicates the tensor product. Suppose that we also have the optimum sensitivity derivatives up to second order. Our goal is to estimate the mean $\bar{\boldsymbol{x}}^*$ and covariance $\boldsymbol{R}^{x^*}$ of system parameters. With a Taylor series expansion, we can express the estimated system parameter vector as

$$\boldsymbol{x}^*(\hat{\boldsymbol{\Phi}}) = \boldsymbol{x}^*(\bar{\boldsymbol{\Phi}}) + \sum_{\mu=1}^{N_L N_M} \boldsymbol{x}^*_{,\mu}(\bar{\boldsymbol{\Phi}})(\hat{\Phi}_\mu - \bar{\Phi}_\mu) + \frac{1}{2} \sum_{\mu=1}^{N_L N_M} \sum_{\nu=1}^{N_L N_M} \boldsymbol{x}^*_{,\mu\nu}(\bar{\boldsymbol{\Phi}})(\hat{\Phi}_\mu - \bar{\Phi}_\mu)(\hat{\Phi}_\nu - \bar{\Phi}_\nu) + \mathcal{O}^3 \tag{19}$$

Equation (19) is valid unless the set of the active constraints changes. With these definitions, the statistical indices of the system parameters are approximated as

$$\bar{\boldsymbol{x}}^* \equiv E[\boldsymbol{x}^*(\hat{\boldsymbol{\Phi}})] = \boldsymbol{x}^*(\bar{\boldsymbol{\Phi}}) + \frac{1}{2} \sum_{\mu=1}^{N_L N_M} \sum_{\nu=1}^{N_L N_M} \boldsymbol{x}^*_{,\mu\nu}(\bar{\boldsymbol{\Phi}}) R^\Phi_{\mu\nu} + \mathcal{O}^3 \tag{20}$$

$$\boldsymbol{R}^{x^*} \equiv E[(\boldsymbol{x}^*(\hat{\boldsymbol{\Phi}}) - \bar{\boldsymbol{x}}^*) \otimes (\boldsymbol{x}^*(\hat{\boldsymbol{\Phi}}) - \bar{\boldsymbol{x}}^*)] = \sum_{\mu=1}^{N_L N_M} \sum_{\nu=1}^{N_L N_M} \boldsymbol{x}^*_{,\mu}(\bar{\boldsymbol{\Phi}}) \otimes \boldsymbol{x}^*_{,\nu}(\bar{\boldsymbol{\Phi}}) R^\Phi_{\mu\nu} + \mathcal{O}^3 \tag{21}$$

With Eqs. (20) and (21), we can obtain the statistical indices of the system parameters as shown in Fig. 2a.

As a simple example of uncertain measurement distributions, let us consider the proportional error given by

$$\hat{\Phi}_\mu = \bar{\Phi}_\mu (1 + \zeta_\mu) \tag{22}$$

where ζ_μ is a uniform random variate in the range $[-a, a]$ (Refs. 3, 14). With these simplifications, Eqs. (20) and (21) lead to the following formula:

$$\bar{\boldsymbol{x}}^*(a) = \boldsymbol{x}^*(\bar{\boldsymbol{\Phi}}) + \frac{a^2}{6} \sum_{\mu=1}^{N_L N_M} \boldsymbol{x}^*_{,\mu\mu}(\bar{\boldsymbol{\Phi}}) \bar{\Phi}^2_\mu + \mathcal{O}^4 \tag{23}$$

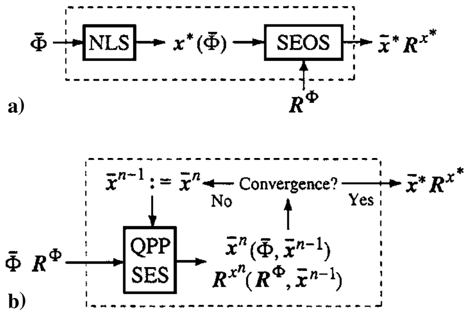


Fig. 2 Basic concepts of a) optimum sensitivity based method and b) conventional sensitivity based methods [statistical evaluation based on optimum sensitivity (SEOS), quadratic programming problem (QPP), statistical evaluation based on conventional sensitivity (SES)].

$$R^{x^*}(a) = \frac{a^2}{3}\sum_{\mu=1}^{N_L N_M} x^*_{,\mu}(\bar{\Phi}) \otimes x^*_{,\mu}(\bar{\Phi})\bar{\Phi}^2_\mu + \mathcal{O}^4 \tag{24}$$

In Sec. IV, we will use the amplitude a to quantify the level of the uncertainty present in the measurements. In addition, we assume that ω_m and $\bar{\Phi}$ are given as the solution of the eigenvalue problem Eq. (1) for the specified nominal value x^N regardless of the value of a, unless otherwise specified. Note, however, that it may be difficult to justify this assumption when a is large because many factors contribute to the uncertainty in the observation. Although we have to be careful in such a case, we employed this assumption because it is simple, it has been used frequently in the literature, and it leads to some interesting observations.

The following remarks can be made:

1) Equation (20) shows that both the second-order optimum sensitivity derivatives and the covariance of the observation have the dominant effect on the estimated mean of the system parameters.

2) With Eq. (20), we can assess how the estimation bias is likely to manifest.

3) From Eq. (21) we can see that the first-order sensitivity derivatives, along with the covariance of the measurements, influence the covariance of the system parameters the most.

4) The third-order terms are exactly zero in Eqs. (23) and (24).

5) The sensitivities $x^*_{,\mu\nu}$ do not appear in Eq. (23) for $\mu \neq \nu$. Hence, they need not be computed in this case.

V. Comparisons with the Conventional Sensitivity-Based Methods

In this section, we show that the optimum sensitivity-based method can be viewed as a generalization of the conventional sensitivity-based methods, wherein conventional sensitivity (i.e., the sensitivity of system response with respect to the system parameters) is used to obtain the mean and covariance of the system parameters. Link[13] gave a concise explanation of how one can use the conventional sensitivity information in two conventional sensitivity-based methods, the classical weighted least-squares method and the extended least-squares method. The classical least-squares method uses no penalty function, whereas the extended least-squares method uses the penalty function defined by the increment of the system parameters and the estimate of the covariance of the system parameters specified before optimization. Similar procedures are used to obtain the statistical indices of the system parameters in the minimum variance method, where the estimate of the covariance of the system parameters is updated at each iteration in the optimization process.[4,16]

Let us derive the formulation for parameter estimation from modal response using conventional sensitivity information. We omit the inequality constraints in this section to simplify the discussion, although it is straightforward to incorporate them. In the conventional sensitivity-based methods, nonlinear least-squares minimization is performed only once for the mean of the observations $\bar{\Phi}$ as shown in Fig. 2b. Let a superscript $n(=1, 2, 3, \ldots)$ indicate that the variable is evaluated at the nth iteration in the nonlinear optimization process. Let us suppose that the mean of the system parameters $\bar{x}^{n-1}$ is known at the $(n-1)$th iteration. The subproblem at the nth iteration is, then, to find $\Delta x^n \equiv x^n - \bar{x}^{n-1}$ that minimizes the objective function J^1_L defined hereafter. With the first-order Taylor series expansion about $\bar{x}^{n-1}$, the linearized error function e^1_{Lm} can be defined as

$$e^1_{Lm}\left(\Delta x^n, \bar{\phi}_m\right) \equiv e_m\left(\bar{x}^{n-1}, \bar{\phi}_m\right) + \nabla e_m\left(\bar{x}^{n-1}, \bar{\phi}_m\right)\Delta x^n \tag{25}$$

With this linearized error function, we can define the objective function J^1_L as

$$J^1_L(\Delta x^n, \bar{\Phi}) \equiv \frac{1}{2}\sum_{m=1}^{N_M}\alpha_m e^1_{Lm}\left(\Delta x^n, \bar{\phi}_m\right)\cdot e^1_{Lm}\left(\Delta x^n, \bar{\phi}_m\right) \tag{26}$$

The increment Δx^n is obtained by solving the simultaneous linear equation

$$\frac{\partial J^1_L\left(\Delta x^n, \bar{\phi}_m\right)}{\partial \Delta x^n} = H_1(\bar{x}^{n-1}, \bar{\Phi})\Delta x^n + C^0(\bar{x}^{n-1}, \bar{\Phi}) = \mathbf{0} \tag{27}$$

where

$$[H_1]_{ij} \equiv \sum_{m=1}^{N_M}\alpha_m \nabla_i e_m \cdot \nabla_j e_m = \nabla^2_{ij} J - \sum_{m=1}^{N_M}\alpha_m e_m \cdot \nabla^2_{ij} e_m \tag{28}$$

$$[C^0]_i \equiv \sum_{m=1}^{N_M}\alpha_m \nabla_i e_m \cdot e_m = \nabla_i J \tag{29}$$

Observe that Eq. (28) corresponds to the Gauss–Newton approximation of the Hessian of J, where the error function is approximated as $e_m = \mathbf{0}$.

Next, consider the statistical indices of the estimate at each iteration. In the conventional sensitivity-based methods, the original error function e_m is often defined such that e_m is linear with respect to $\hat{\Phi}$, that is, e_m has no cross terms between x and $\hat{\Phi}$. Nonetheless, the output error estimator defined by Eq. (5) has this cross term. A modification is, therefore, needed to obtain the statistical indices of the system parameters based on conventional sensitivity. For this purpose, we linearize the error function e_m with respect not only to x but also to $\hat{\Phi}$. Let us define another linear error function e^2_{Lm} as

$$e^2_{Lm}\left(\Delta x^n, \Delta\hat{\phi}_m\right) \equiv e^1_{Lm}\left(\Delta x^n, \bar{\phi}_m\right) + \sum_{\mu=1}^{N_L N_M} e_{m,\mu}\left(\bar{x}^{n-1}, \bar{\phi}_m\right)\Delta\hat{\Phi}_\mu \tag{30}$$

where $\Delta\hat{\Phi} \equiv \hat{\Phi} - \bar{\Phi}$. Let us define a new objective function J^2_L as

$$J^2_L(\Delta x^n, \Delta\hat{\Phi}) \equiv \frac{1}{2}\sum_{m=1}^{N_M}\alpha_m e^2_{Lm}\left(\Delta x^n, \Delta\hat{\phi}_m\right)\cdot e^2_{Lm}\left(\Delta x^n, \Delta\hat{\phi}_m\right) \tag{31}$$

We obtain Δx^n by solving

$$\frac{\partial J^2_L(\Delta x^n, \Delta\hat{\Phi})}{\partial \Delta x^n} = H_1(\bar{x}^{n-1}, \bar{\Phi})\Delta x^n + C^0(\bar{x}^{n-1}, \bar{\Phi}) + H_2(\bar{x}^{n-1}, \bar{\Phi})\Delta\hat{\Phi} = \mathbf{0} \tag{32}$$

where the $i\mu$th element of H_2 is defined as

$$[H_2]_{i\mu} \equiv \sum_{m=1}^{N_M}\alpha_m \nabla_i e_m \cdot e_{m,\mu} = \nabla_i J_{,\mu} - \sum_{m=1}^{N_M}\alpha_m \nabla_i e_{m,\mu} \cdot e_m \tag{33}$$

Notice that H_2 can be regarded as the Gauss–Newton approximation of $\nabla J_{,\mu}$. The solution of Eq. (32) is obtained as

$$\Delta x^n = -H_1^{-1}C^0 - H_1^{-1}H_2\Delta\hat{\Phi} \tag{34}$$

where $\boldsymbol{H}_1^{-1}$ indicates a generalized inverse of $\boldsymbol{H}_1$. Noticing that $E[\Delta\hat{\boldsymbol{\Phi}}]=\boldsymbol{0}$ and $E[\Delta\hat{\boldsymbol{\Phi}}\otimes\Delta\hat{\boldsymbol{\Phi}}]=\boldsymbol{R}^{\Phi}$, we obtain the statistical indices of $\boldsymbol{x}^n$ as

$$\bar{\boldsymbol{x}}^n \equiv E[\bar{\boldsymbol{x}}^{n-1}+\Delta\boldsymbol{x}^n]=\bar{\boldsymbol{x}}^{n-1}-\boldsymbol{H}_1^{-1}\boldsymbol{C}^0 \tag{35}$$

$$\boldsymbol{R}^{x^n} \equiv E[(\boldsymbol{x}^n-\bar{\boldsymbol{x}}^n)\otimes(\boldsymbol{x}^n-\bar{\boldsymbol{x}}^n)]=\boldsymbol{H}_1^{-1}\boldsymbol{H}_2\boldsymbol{R}^{\Phi}\boldsymbol{H}_2^T\boldsymbol{H}_1^{-T} \tag{36}$$

At the optimum solution, Eq. (36) yields the covariance of $\boldsymbol{x}^*$.

The following remarks can be made:

1) In the conventional sensitivity-based method, there is no need to evaluate $\boldsymbol{R}^{x^n}$ at each iteration to obtain $\boldsymbol{R}^{x^*}$ because only $\bar{\boldsymbol{x}}^{n-1}$ is needed to update $\boldsymbol{x}^n$ and $\boldsymbol{R}^{x^n}$.

2) With the Gauss–Newton approximation, the first-order optimum sensitivity derivatives computed by Eq. (21) reduce to $-\boldsymbol{H}_1^{-1}\boldsymbol{H}_2$ in Eq. (34) evaluated at $\boldsymbol{x}=\bar{\boldsymbol{x}}^*$. Note here that $\boldsymbol{C}^0=\boldsymbol{0}$ at $\boldsymbol{x}=\bar{\boldsymbol{x}}^*$ and that $\boldsymbol{H}_1$ and $\boldsymbol{H}_2$ are the Gauss–Newton approximations of $\nabla^2 J$ and $\nabla J_{,\mu}$, respectively.

3) Equations (20) and (35) show that the conventional sensitivity-based methods provide no information on the bias of an estimate due to the nonlinearities in the error functions.

4) Compared to the conventional sensitivity-based methods, the present method requires additional computations of higher-order derivatives, for example, $\nabla^2 J$ and $\nabla^3 J$. Nonetheless, because these additional computations need to be performed only once, the present method is usually much more efficient than the Monte Carlo simulation.

5) One can recover the mean and covariance of the conventional sensitivity-based methods by neglecting corresponding higher-order derivatives and by applying the Gauss–Newton approximations in the optimum sensitivity-based method.

VI. Example Problems

In this section, we examine the performance and limitation of the present method along with the Monte Carlo simulation. In Sec. IV.A, we derive closed-form expressions of the statistical indices for the two-DOF spring–mass system shown in Fig. 3a. Then, for this system, we compare three sets of the statistical indices obtained by the closed-form solution, the present method, and the Monte Carlo simulation. Note that for more general and complex models, it would be very difficult, if not impossible, to derive such a closed-form solution. Hence, the two-DOF example provides a valuable opportunity to evaluate these methods. In Sec. VI.B, we apply the present method and Monte Carlo simulation to the six-DOF spring–mass system depicted in Fig. 3b to demonstrate the more general case.

A. Two-DOF Spring–Mass System

Let us estimate the statistical indices of the stiffness parameters of the two-DOF spring–mass system shown in Fig. 3a. We assume that the natural frequency and the mode shape are measured only for the first mode at both DOF ($N_M=1$ and $N_L=2$). We consider only the lower bound $\boldsymbol{c}_L=\boldsymbol{0}$. The eigenvalue problem for the first mode can be written as

$$\begin{bmatrix} x_1+x_2 & -x_2 \\ -x_2 & x_2 \end{bmatrix}\begin{bmatrix}\hat{\Phi}_1\\ \hat{\Phi}_2\end{bmatrix}=\omega_1^2\begin{bmatrix} m_1 & 0\\ 0 & m_2\end{bmatrix}\begin{bmatrix}\hat{\Phi}_1\\ \hat{\Phi}_2\end{bmatrix} \tag{37}$$

Noting that the left-hand side of Eq. (37) is a linear combination of x_1 and x_2, we can rewrite Eq. (37) as

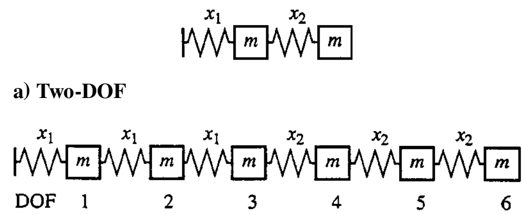
$$\begin{bmatrix}\hat{\Phi}_1 & \hat{\Phi}_1-\hat{\Phi}_2\\ 0 & -\hat{\Phi}_1+\hat{\Phi}_2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\omega_1^2\begin{bmatrix} m_1 & 0\\ 0 & m_2\end{bmatrix}\begin{bmatrix}\hat{\Phi}_1\\ \hat{\Phi}_2\end{bmatrix} \tag{38}$$

Assume that $\hat{\Phi}_1\neq 0$ and $\hat{\Phi}_1\neq\hat{\Phi}_2$. Then the estimate $\boldsymbol{x}^*$ can be obtained uniquely as

$$x_1^*=\omega_1^2[m_1+m_2(\hat{\Phi}_2/\hat{\Phi}_1)] \tag{39a}$$

$$x_2^*=\omega_1^2 m_2[\hat{\Phi}_2/(\hat{\Phi}_2-\hat{\Phi}_1)] \tag{39b}$$

Observe that the estimate $\boldsymbol{x}^*$ is a function of $\hat{\Phi}_1$ and $\hat{\Phi}_2$.



b) Six-DOF spring–mass systems

Fig. 3 Example models.

With Eq. (39), we can derive the closed-form expressions of the exact and approximate statistical indices of $\boldsymbol{x}^*$. First, consider the exact statistical indices. Define $A_i(a)$ as

$$A_1\equiv\frac{\bar{\Phi}_1+\bar{\Phi}_2}{\bar{\Phi}_1-\bar{\Phi}_2},\qquad A_2(a)\equiv\frac{1}{2a}\log\frac{1+a}{1-a}$$

$$A_3(a)\equiv\frac{1}{\left(A_1^2-1\right)a^2}\log\frac{1-a^2}{1-A_1^2a^2}$$

$$A_4(a)\equiv\frac{1}{2A_1a}\log\frac{1+A_1a}{1-A_1a},\qquad A_5(a)\equiv\log\frac{1-a^2}{1-A_1^2a^2} \tag{40}$$

Recall that a is the level of the uncertainty present in measurements defined in Sec. IV. Performing integration in accordance with the standard definition of mean and covariance of continuous functions, we obtain the exact statistical indices as follows:

$$\bar{x}_1^*(a)=\omega_1^2\left[m_1+m_2\frac{\bar{\Phi}_2}{\bar{\Phi}_1}A_2(a)\right] \tag{41a}$$

$$\bar{x}_2^*(a)=\frac{\omega_1^2m_2}{2}\left\{1+\left(\frac{\bar{\Phi}_1}{\bar{\Phi}_2}-\frac{\bar{\Phi}_2}{\bar{\Phi}_1}\right)\left[\frac{A_5(a)+\left(A_1^2-1\right)A_3(a)}{4}+A_2(a)\right]-\left(\frac{\bar{\Phi}_1}{\bar{\Phi}_2}+\frac{\bar{\Phi}_2}{\bar{\Phi}_1}\right)A_1A_4(a)\right\} \tag{41b}$$

$$R_{11}^x(a)=\omega_1^4\left(m_1^2+2m_1m_2\frac{\bar{\Phi}_2}{\bar{\Phi}_1}A_2(a)-\frac{1}{3}m_2^2\frac{\bar{\Phi}_2^2}{\bar{\Phi}_1^2}\frac{a^2+3}{a^2-1}\right)-\bar{x}_1^{*2}(a) \tag{41c}$$

$$R_{22}^x(a)=\omega_1^4m_2^2\left\{1+\frac{\bar{\Phi}_1}{\bar{\Phi}_2}\left[\frac{A_5(a)+\left(A_1^2-1\right)A_3(a)}{4}+A_2(a)-A_1A_4(a)\right]\right\}-\bar{x}_2^{*2}(a) \tag{41d}$$

$$R_{12}^x(a)=R_{21}^x(a)=\omega_1^2\left[(m_1+m_2)\bar{x}_2^*(a)+\omega_1^2m_2^2\frac{\bar{\Phi}_2}{\bar{\Phi}_1}A_2(a)\right]-\bar{x}_1^*(a)\bar{x}_2^*(a) \tag{41e}$$

As shown in Appendix B, we have the following relations:

$$\lim_{a\to 0}A_2(a)=\lim_{a\to 0}A_3(a)=\lim_{a\to 0}A_4(a)=1,\qquad \lim_{a\to 0}A_5(a)=0 \tag{42}$$

With these limits, we can show that

$$\lim_{a\to 0}\bar{\boldsymbol{x}}^*(a)=\boldsymbol{x}^*(\bar{\boldsymbol{\Phi}}),\qquad \lim_{a\to 0}\boldsymbol{R}^{x^*}(a)=\boldsymbol{0} \tag{43}$$

Next, let us consider the approximate statistical indices using optimum sensitivity. Because we have the closed-form expressions of $\boldsymbol{x}^*(\hat{\boldsymbol{\Phi}})$, it is straightforward to obtain the optimum sensitivity derivatives. Differentiating $\boldsymbol{x}^*$ with respect to $\hat{\boldsymbol{\Phi}}$ and substituting the derivatives into formula (23) and (24), we have

$$\bar{x}_1^*(a) = \omega_1^2\left(m_1 + m_2\frac{\bar{\Phi}_2}{\bar{\Phi}_1}\right) + \frac{\omega_1^2 m_2}{3}\frac{\bar{\Phi}_2}{\bar{\Phi}_1}a^2 + \mathcal{O}^4$$

$$\bar{x}_2^*(a) = \omega_1^2 m_2\frac{\bar{\Phi}_2}{\bar{\Phi}_2 - \bar{\Phi}_1} + \omega_1^2 m_2\frac{\bar{\Phi}_1\bar{\Phi}_2(\bar{\Phi}_1 + \bar{\Phi}_2)}{3(\bar{\Phi}_2 - \bar{\Phi}_1)^3}a^2 + \mathcal{O}^4$$

$$R_{11}^x(a) = \frac{2}{3}\omega_1^4 m_2^2\frac{\bar{\Phi}_2^2}{\bar{\Phi}_1^2}a^2 + \mathcal{O}^4$$

$$R_{22}^x(a) = \frac{2}{3}\omega_1^4 m_2^2\frac{\bar{\Phi}_1^2\bar{\Phi}_2^2}{(\bar{\Phi}_2 - \bar{\Phi}_1)^4}a^2 + \mathcal{O}^4$$

$$R_{12}^x(a) = R_{21}^x(a) = -\frac{2}{3}\omega_1^4 m_2^2\frac{\bar{\Phi}_2^2}{(\bar{\Phi}_2 - \bar{\Phi}_1)^2}a^2 + \mathcal{O}^4 \qquad (44)$$

Based on the preceding equations, we can compare numerically the $\bar{\boldsymbol{x}}^*$–a relation, or the relation between the estimated mean of the system parameters and the level of uncertainty in the measurement, obtained by the exact expressions [Eq. (41)], the present method [Eq. (44)], and the Monte Carlo simulation method. The numerical data were taken to be $m = 1.00$ and $\boldsymbol{x}^N = \{2.16, 1.44\}^T$. For the nominal stiffness $\boldsymbol{x}^N$, the eigenvalue and eigenvectors were computed as $\omega_1^2 = 0.72$ and $\bar{\boldsymbol{\Phi}}^T = \{0.50, 1.00\}$. In the Monte Carlo simulation, nonlinear optimization was preformed 5000 times from the same starting point $\bar{\boldsymbol{x}}^0 = \{1.8, 1.8\}^T$. The details of the nonlinear optimization technique used in this simulation were documented by Banan and Hjelmstad.[3] The comparison of the $\bar{\boldsymbol{x}}^*$–a relations is shown in Fig. 4. We can make the following observations from Fig. 4:

1) There is a critical value $a = a_{cr}$ at which $\bar{x}_2^*$ blows up. Noticing that there is a logarithm function $\log(1 + A_1 a)$ in Eq. (41b), we can obtain a_{cr} as

$$a_{cr} = -\frac{1}{A_1} = -\frac{\bar{\Phi}_1 - \bar{\Phi}_2}{\bar{\Phi}_1 + \bar{\Phi}_2} = \frac{1}{3} \qquad (45)$$

Observe that it is possible for $\hat{\Phi}_1$ to be equal or close to $\hat{\Phi}_2$ when $a \geq a_{cr}$ and that x_2^* blows up when $\hat{\Phi}_1 \approx \hat{\Phi}_2$ [see Eq. (39b)].

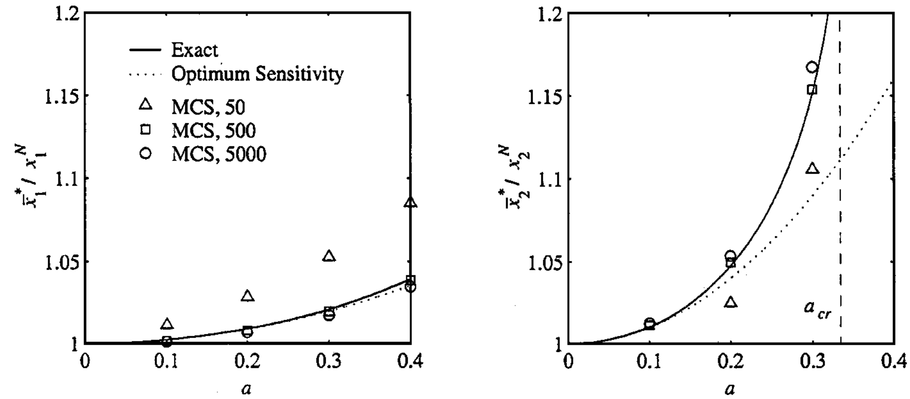
2) The truncation error of the present method in $\bar{x}_2^*$ is significant for $a > 0.2$, whereas it is small for $a \leq 0.2$. In contrast, the error in $\bar{x}_1^*$ is small for the entire range shown in Fig. 4. Nevertheless, note that $\bar{x}_1^*$ blows up at $a = 1$ because it has the term of $\log(1 - a)$ as shown in Eq. (41a). Apparently, the truncation error in $\bar{x}_1^*$ becomes significant as $a \to 1$.

3) The Monte Carlo simulation succeeded in capturing the blowup. The Monte Carlo simulation gave estimates close to the exact expressions except the estimate of $\bar{x}_2^*$ for $a = 0.4(> a_{cr} = \frac{1}{3})$, and, when a was 0.4, $\bar{x}_2^*$ was much larger than in the other cases, as shown in Figs. 5 and 6.

4) The estimates of the two system parameters lie along a definite curve in Fig. 6, implying a nonlinear constraint in uncertainty of the estimates. In this case, the probability distribution of the system parameters was not Gaussian in the two-dimensional parameter space. Hence, the mean and covariance did not adequately describe the probability distribution of the parameters. These problems are, in some sense, pathological. They seem to imply an inherent insufficiency of the data to represent both parameters. No algorithm can cure insufficient information. A similar observation can be made when the nonlinear optimization problem has multiple solutions.

B. Six-DOF Spring–Mass System

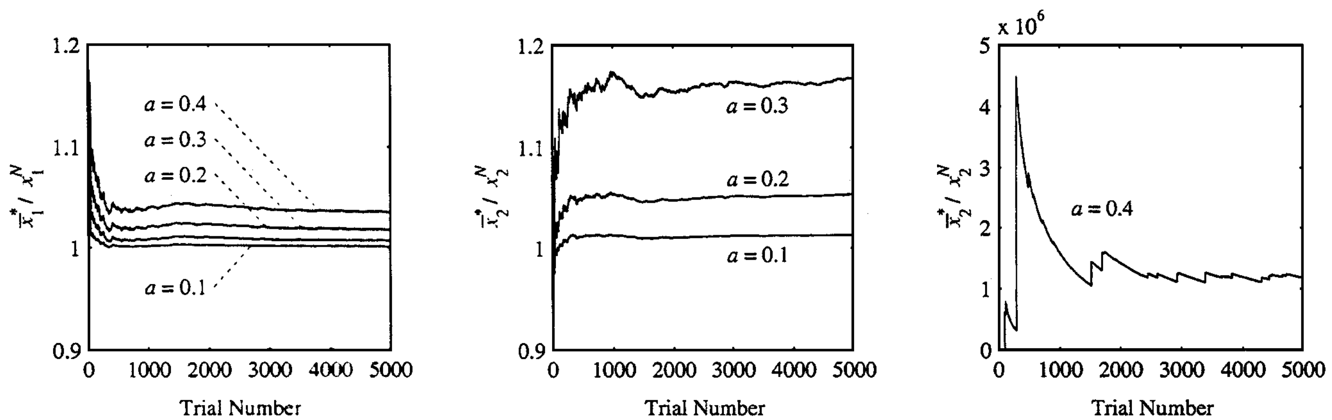
Consider the parameter estimation of the six-DOF spring–mass system shown in Fig. 3b. We assume that the observations were made only for the first mode at the third, fourth, fifth, and sixth DOF ($N_L = 4$, $N_M = 1$). The structural parameters and the



a) $\bar{x}_1^*/x_1^N$–a relation

b) $\bar{x}_2^*/x_2^N$–a relation

Fig. 4 Comparison of relations between estimated mean $\bar{x}^*$ of system parameter and level a of uncertain measurement for two-DOF system.



a) $\bar{x}_1^*/x_1^N$–trial number relation for $a = 0.1$–0.4

b) $\bar{x}_2^*/x_2^N$–trial number relation for $a = 0.1$–0.3

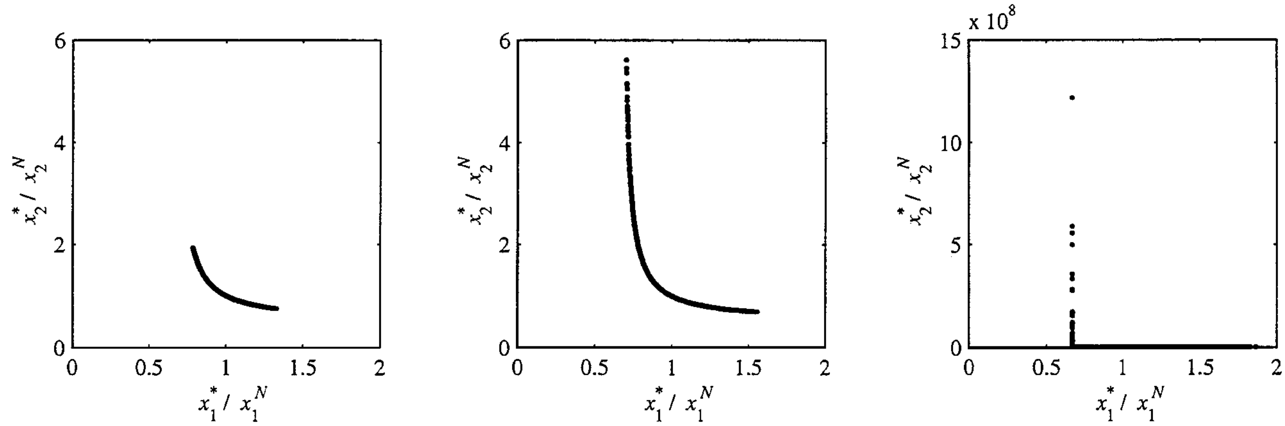
c) $\bar{x}_2^*/x_2^N$–trial number relation for $a = 0.4$

Fig. 5 Variation of mean of estimates with respect to trial number in MCS for two-DOF system.

parameters for the Monte Carlo simulation were the same as those for the two-DOF system. For the nominal stiffness, we had $\omega_1^2 = 0.1127$ and $\hat{\Phi}^T = \{0.5605, 0.7713, 0.9217, 1.0000\}$. For this system, Hjelmstad[17] showed that, when there was no uncertainty in the measurement, the estimate had only one large and another small attraction basin when solving the nonlinear least-squares minimization from random starting points. Hence, we can expect that the problem of uniqueness of estimates[17,18] has little effect on the parameter estimation of this system.

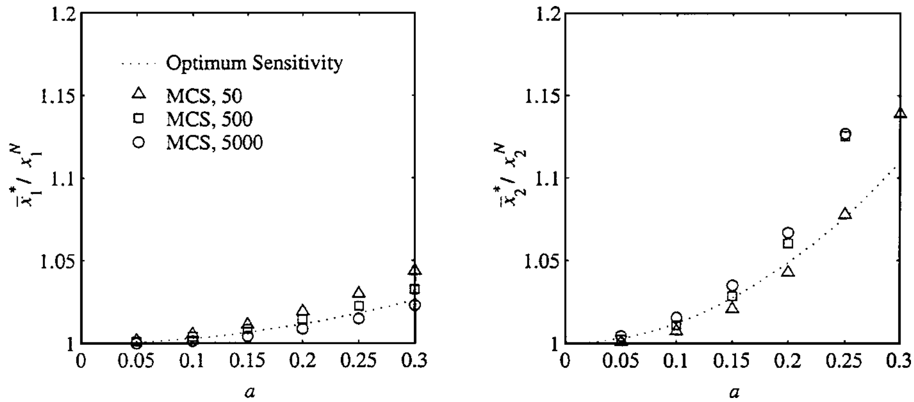
Figures 7–9 depict the results of the present method and the Monte Carlo simulation. For $\bar{x}_1^*$, the result of the Monte Carlo simulation approached that of the present method as the number of trials increased. On the other hand, for $\bar{x}_2^*$, the difference between these results became significant for $a \geq 0.2$. When a was 0.3, the value of $\bar{x}_2^*$ calculated by the Monte Carlo simulation did not converge after 5000 trials, whereas it converged in fewer than 1000 trials in the other cases as shown in Fig. 8. From these results and the discussion in Sec. VI. A, we may conclude that $\bar{x}_2^*$ blows up somewhere between $a = 0.25$ and 0.30.

In addition, let us examine the effect of the Gauss–Newton approximation in Eqs. (12) and (13) when $\boldsymbol{e}_m \neq \boldsymbol{0}$. For the measurement in which only $\hat{\Phi}_4$ is changed from 1.0 to 0.8, we obtained $\boldsymbol{x}_{,4}^*$



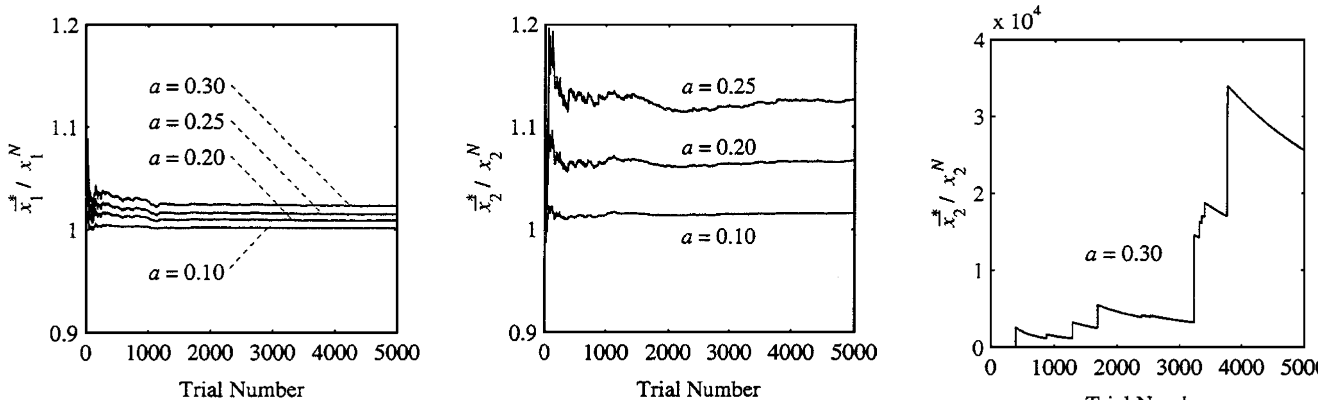
a) $a = 0.2$ b) $a = 0.3$ c) $a = 0.4$

Fig. 6 Scatter of estimates in the MCS for two-DOF system.



a) $\bar{x}_1^*/x_1^N$–a relation b) $\bar{x}_2^*/x_2^N$–a relation

Fig. 7 Comparison of relations between estimated mean $\bar{x}^*$ of system parameter and level a of uncertain measurement for six-DOF system.



a) $\bar{x}_1^*/x_1^N$–trial number relation for $a = 0.10$–0.30 b) $\bar{x}_2^*/x_2^N$–trial number relation for $a = 0.10$–0.25 c) $\bar{x}_2^*/x_2^N$–trial number relation for $a = 0.30$

Fig. 8 Variation of mean of estimates with respect to trial number in MCS for six-DOF system.

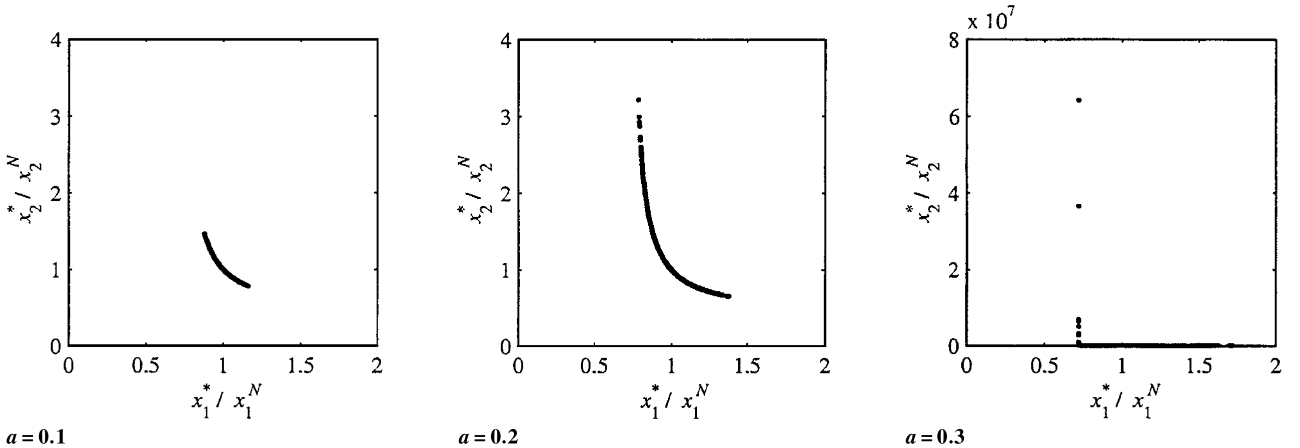


Fig. 9 Scatter of estimates in MCS for six-DOF system.

by using the following three different methods: 1) the finite difference method with $\Delta\hat{\Phi}_4 = 0.001$, 2) direct differentiation (the present method), and 3) direct differentiation with the Gauss–Newton approximation (the conventional sensitivity-based methods). The values of $\boldsymbol{x}_4^{*T}$ computed by these method were as follows: 1) {0.98684, −3.36169}, 2) {0.98684, −3.36168}, and 3) {1.02166, −3.49273}. We can clearly observe the good agreement between the results computed by the present method and the finite difference method. In contrast, the results calculated with the Gauss–Newton approximation do not agree with those of the finite difference method. It is obvious that the error due to the Gauss–Newton approximation becomes large as the norm of residual vectors increases.

VII. Conclusions

We have presented a method for statistical parameter estimation of a structure from its modal response based on the concept of optimum sensitivity. With the present method, we can assess the bias of an estimate due to nonlinearities in the parametric model. This is the primary difference between the present method and the previous methods of statistical parameter estimation that use conventional sensitivity information. In addition, the present method preserves the efficiency compared with the Monte Carlo simulation, although the Monte Carlo simulation is more robust. Careful examination of the formulations of the present (optimum sensitivity-based) and the previous (conventional sensitivity-based) methods has revealed that we can view the conventional sensitivity-based methods as a specialization of the present method.

In the example problems, we have applied the present method and Monte Carlo simulation to two- and six-DOF spring–mass systems. In addition, we have derived closed-form expressions of the statistical indices of system parameters for the two-DOF system. With these closed-form expressions, we have found that the covariance of the measurement has a critical level at which the statistical indices of an estimate blows up. It appears that such a sudden blowup has not been reported in the literature but might, indeed, be a common phenomenon in parameter estimation problems. Although the present method failed to obtain a reliable estimate in this case, it provided results close to the closed-form expressions for a wide range of uncertain levels of the measurement. On the other hand, Monte Carlo simulation gave results that were consistent with the closed-form expressions even when the blowup took place. In the six-DOF system problem, for which such closed-form expressions are unlikely, we observed that the present method provided results consistent with those of Monte Carlo simulation, except when the estimates were prone to blow up.

Based on the results of this research, we suggest that the mixed use of the present method and the Monte Carlo simulation would be effective in terms of efficiency and reliability to many practical problems of statistical parameter estimation. Although we have treated only parameter estimation from modal response and have not used regularization methods, it is possible to apply the present method to other types of parameter estimation and to incorporate the regularization methods without significant modifications. Because optimum sensitivity provides much information on the statistics of estimates, applications of optimum sensitivity to the problem of finding the optimum sensor location or the optimum weighting factor of nonlinear least-squares estimators are promising future research topics. In the field of stochastic mechanics, several techniques have been proposed that give more accurate statistical indices than the perturbation method based on the conventional sensitivity information. It would be interesting to apply such techniques to statistical parameter estimation based on the concept of optimum sensitivity.

Appendix A: Derivatives of Error Function

The derivatives of the error function are obtained as follows:

$$\nabla_i \boldsymbol{e}_m = -\omega_m^2 \boldsymbol{Q} \nabla_i \boldsymbol{B}_m^{-1} \hat{\boldsymbol{M}} \hat{\boldsymbol{\phi}}_m, \qquad \nabla_{ij}^2 \boldsymbol{e}_m = -\omega_m^2 \boldsymbol{Q} \nabla_{ij}^2 \boldsymbol{B}_m^{-1} \hat{\boldsymbol{M}} \hat{\boldsymbol{\phi}}_m$$

$$\nabla_{ijk}^3 \boldsymbol{e}_m = -\omega_m^2 \boldsymbol{Q} \nabla_{ijk}^3 \boldsymbol{B}_m^{-1} \hat{\boldsymbol{M}} \hat{\boldsymbol{\phi}}_m, \qquad \boldsymbol{e}_{m,\mu} = \boldsymbol{q}_{m\mu} - \omega_m^2 \boldsymbol{Q} \boldsymbol{B}_m^{-1} \hat{\boldsymbol{M}} \boldsymbol{q}_{m\mu}$$

$$\nabla_i \boldsymbol{e}_{m,\mu} = -\omega_m^2 \boldsymbol{Q} \nabla_i \boldsymbol{B}_m^{-1} \hat{\boldsymbol{M}} \boldsymbol{q}_{m\mu}$$

$$\nabla_{ij}^2 \boldsymbol{e}_{m,\mu} = -\omega_m^2 \boldsymbol{Q} \nabla_{ij}^2 \boldsymbol{B}_m^{-1} \hat{\boldsymbol{M}} \boldsymbol{q}_{m\mu}, \qquad \boldsymbol{e}_{m,\mu\nu} = \nabla_i \boldsymbol{e}_{m,\mu\nu} = \boldsymbol{0} \tag{A1}$$

Here, $\boldsymbol{q}_{m\mu}$ is a boolean vector defined as $\boldsymbol{q}_{m\mu} \equiv \hat{\phi}_{m,\mu}$. With $\nabla_i \boldsymbol{B}_m = \nabla_i \boldsymbol{K}$, the derivatives of $\boldsymbol{B}_m^{-1}$ can be obtained recursively as

$$\nabla_i \boldsymbol{B}_m^{-1} = -\boldsymbol{B}_m^{-1} \nabla_i \boldsymbol{B}_m \boldsymbol{B}_m^{-1}$$

$$\nabla_{ij}^2 \boldsymbol{B}_m^{-1} = -\nabla_j \boldsymbol{B}_m^{-1} \nabla_i \boldsymbol{B}_m \boldsymbol{B}_m^{-1} - \boldsymbol{B}_m^{-1} \nabla_i \boldsymbol{B}_m \nabla_j \boldsymbol{B}_m^{-1}$$

$$\nabla_{ijk}^3 \boldsymbol{B}_m^{-1} = -\nabla_{jk}^2 \boldsymbol{B}_m^{-1} \nabla_i \boldsymbol{B}_m \boldsymbol{B}_m^{-1} - \nabla_j \boldsymbol{B}_m^{-1} \nabla_i \boldsymbol{B}_m \nabla_k \boldsymbol{B}_m^{-1}$$

$$- \nabla_k \boldsymbol{B}_m^{-1} \nabla_i \boldsymbol{B}_m \nabla_j \boldsymbol{B}_m^{-1} - \boldsymbol{B}_m^{-1} \nabla_i \boldsymbol{B}_m \nabla_{jk}^2 \boldsymbol{B}_m^{-1} \tag{A2}$$

Appendix B: Limits of Functions Defined in Eq. (40)

Let us derive Eq. (42). $A_2(a)$ can be expressed as

$$A_2(a) \equiv \frac{1}{2a} \log \frac{1+a}{1-a} = \frac{\log(1+a) - \log(1-a)}{1+a-(1-a)} \tag{B1}$$

Hence, we have the following equality as

$$\lim_{a \to 0} A_2(a) = \left. \frac{\mathrm{d} \log z}{\mathrm{d} z} \right|_{z=1} = \left. \frac{1}{z} \right|_{z=1} = 1 \tag{B2}$$

where z is an auxiliary variable. The limit of $A_3(a)$ and $A_4(a)$ can be obtained similarly. The limit of $A_5(a)$ is obtained as

$$\lim_{a \to 0} A_5(a) = \log 1 = 0 \tag{B3}$$

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